Exercise 44

The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \\ \dfrac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

Solution

$(GM/R^3)r$ is a polynomial, and $GM(1/r^2)$ is a rational function. These are continuous on their respective domains by Theorem 7. Any points of discontinuity, then, can only occur at the endpoints of the intervals that these functions are defined on. Check whether $F(r)$ is continuous at $r = R$.

$$\lim_{r\to R^-} F(r) \stackrel{?}{=} \lim_{r\to R^+} F(r) \stackrel{?}{=} F(R)$$

$$\lim_{r\to R^-} \frac{GMr}{R^3} \stackrel{?}{=} \lim_{r\to R^+} \frac{GM}{r^2} \stackrel{?}{=} \left.\frac{GM}{r^2}\right|_{r=R}$$

$$\frac{GM}{R^3}\left(\lim_{r\to R^-} r\right) \stackrel{?}{=} \frac{GM}{\left(\lim\limits_{r\to R^+} r\right)\left(\lim\limits_{r\to R^+} r\right)} \stackrel{?}{=} \frac{GM}{R^2}$$

$$\frac{GM}{R^3}(R) \stackrel{?}{=} \frac{GM}{(R)(R)} \stackrel{?}{=} \frac{GM}{R^2}$$

$$\frac{GM}{R^2} = \frac{GM}{R^2} = \frac{GM}{R^2}$$

This is a true statement, so $F(r)$ is continuous at $r = R$. Therefore, $F(r)$ is a continuous function of r.